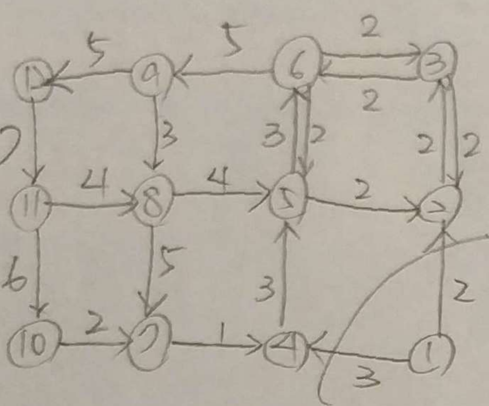
# 台大104數學

1.

$p \to q$ 等價於 $\neg p \vee q$ (A)

| $p$ | $q$ | $\neg p$ | $\neg q$ | $\neg p \to \neg q$ | $q \to p$ | $\neg q \to \neg p$ | $\neg q \to p$ | $p \to q$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |

2.




a. ⑫ → ⑨ → ⑥ → ③ → ② → ① = 2

b. ⑫ → ⑨ → ⑥ → ⑤ → ④ → ① = 3

max flow = $\underline{5}$ #

3.

考慮多項式 $(1+x)^n = \binom{n}{0} \cdot x^0 + \binom{n}{1} \cdot x^1 + \binom{n}{2} \cdot x^2 + \cdots + \binom{n}{n} x^n$

$\sum_{i=0}^{n} \binom{n}{i} 2^i = \binom{n}{0} 2^0 + \binom{n}{1} 2^1 + \binom{n}{2} 2^2 + \cdots + \binom{n}{n} 2^n$

$x$ 代入 2 = $\underline{3^n}$ #

4.

$120 = 2^3 \times 3 \times 5$, $p_1 = 2, p_2 = 3, p_3 = 5$ $\quad \phi(n) = 120(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})$

$= \underline{32}$ #

6.

$$\begin{cases} a_n = 2a_{n-1} + 3^{n-1} \\ a_0 = 2 \end{cases}$$

$(\alpha - 2) = 0$ , $a_n^{(h)} = c_1 \cdot 2^n$

$a_n^{(p)} = 3^n \cdot d_0$

代回，$3^n d_0 - 2 \cdot 3^{n-1} d_0 = 3^{n-1}$

$n=0$, $d_0 - \frac{2}{3} d_0 = \frac{1}{3}$ $\quad d_0 = 1 \quad\quad a_n^{(p)} = 3^n$

$a_n = c_1 \cdot 2^n + 3^n$

代入 $a_0 = c_1 + 1 = 2$ , $c_1 = 1$ $\quad\quad \underline{a_n = 2^n + 3^n}$ #

7. 佈於複數空間，$X、Y = 2\times2$，提內積運算

$f(X, Y) = tr(X \cdot Y)$

(a) $f(X,Y) = f(Y,X)$ 因內積要有共軛交換性

(b) 因內積要符合正定性，計算長度要為正數 ⇒ 不會有複數 entry.

(c) 因內積需符合左、右線性 $\begin{cases} g_Y(X) = f(X,Y) \text{要為線性} \\ h_X(Y) = f(X,Y) \text{要為線性} \end{cases}$

(D)

8.

$A_{4\times5} \times B_{5\times5} = AB_{4\times5}$ $\quad\quad AB = 0$ ，B 相當於 A 的 kernel space

$$\begin{bmatrix} -1 & 1 & 3 & -1 & 0 \\ 3 & -1 & -5 & 1 & -6 \\ 1 & 0 & -1 & 2 & 1 \\ -2 & 1 & 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(A) = 3$$

① 根據維度定理 $dim(V^5) = rank(A) + nullity(A)$ ⇒ $nullity(A) = 2$

② 又得知 $dim(Ker(A)) = 2$

9. $tr(A)$ 相加 $= \underline{8}$ #

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由題目之 $B_1, B_2$ 可知 $B_1 \cdot B_2$ 正交

取 $C_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -2 \end{bmatrix}$, $C_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, 找到一個 $C_3$ 和 $C_1 \cdot C_2$ 正交 orthonormal 即可

$\Rightarrow$ 使 $A_3$ 之 orthonormal $= C_3$

$u_1 = (3, -1, 2, -2)$ $\langle u_1, u_1 \rangle = 18$

$u_2 = (-2, 2, 3, -1)$ $\langle u_2, u_2 \rangle = 18$

$\langle u_3, u_3 \rangle = 90$

$u_3 = (-1, -9, 5, -1) - \frac{18}{18}(3, -1, 2, -2) - \frac{0}{18}(-2, 2, 3, -1) = (-4, -8, 3, 1)$

取 $B_3$ 為 $\frac{1}{3\sqrt{10}}\begin{bmatrix} -4 & -8 \\ 3 & 1 \end{bmatrix}$ #

11. 双線性要符合
$$\begin{cases} ① f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \\ ② f(x, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) \\ ③ f(cx, y) = cf(x, y) \\ ④ f(x, cy) = cf(x, y) \end{cases}$$

$$\begin{bmatrix} f(\beta_1, \beta_1) & f(\beta_1, \beta_2) & f(\beta_1, \beta_3) & f(\beta_1, \beta_4) \\ f(\beta_2, \beta_1) & f(\beta_2, \beta_2) & f(\beta_2, \beta_3) & f(\beta_2, \beta_4) \\ f(\beta_3, \beta_1) & f(\beta_3, \beta_2) & f(\beta_3, \beta_3) & f(\beta_3, \beta_4) \\ f(\beta_4, \beta_1) & f(\beta_4, \beta_2) & f(\beta_4, \beta_3) & f(\beta_4, \beta_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

5. 則 $x_1 \geq 1$ , $x_2 > x_1$ ..... $x_n > x_{n-1}$ , $r \geq x_n$

① 令 $y_1 = x_1 - 1 \geq 0$ , $y_2 = x_2 - x_1 \geq 1$ , ..... $y_n = x_n - x_{n-1} \geq 1$

$y_{n+1} = r - x_n \geq 0$

② $y_1 + y_2 + \cdots + y_n + y_{n+1} = x_1 - 1 + x_2 - x_1 + \cdots + x_n - x_{n-1} + r - x_n$
$= r - 1$

③ 又因 $y_2$、$y_3$ ..... $y_n$ 皆要 $\geq 1$ (共 $n-1$ 個).

④ $y_1 + y_2 + \cdots + y_n + y_{n+1} = r - 1 - (n-1) = r - n$

⑤ $\binom{r-n+n}{r-n} = \underline{\binom{r}{n}}$ #