107 台大數學

1. $A \cdot B \in M_{n\times n}$，佈於 $\mathbb{C}$ or $\mathbb{R}$

(A) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$ ，$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B$ $\begin{bmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix} = AB$

$\begin{bmatrix} b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22} \\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22} \end{bmatrix} = BA$

則 $tr(AB) = a_{11}b_{11} + a_{12}b_{12} = tr(BA)$ 成立.

(B) $tr(A)tr(B) = (a_{11}+a_{22})(b_{11}+b_{22}) = a_{11}b_{11} + a_{11}b_{22} + a_{22}b_{11} + a_{22}b_{22}$ (不一定成立)

$\neq a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{12} + a_{22}b_{22} = tr(AB)$

(C) $tr(B^{-1}AB) = tr(ABB^{-1}) = tr(AI) = tr(A)$

(D) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2+a_{12}a_{21} & a_{11}a_{12}+a_{12}a_{22} \\ a_{21}a_{11}+a_{22}a_{21} & a_{21}a_{12}+a_{22}^2 \end{bmatrix} = A^2$

$tr(A^2) = a_{11}^2 + a_{12}a_{21} + a_{21}a_{12} + a_{22}^2 \neq [tr(A)]^2 = a_{11}^2 + 2a_{11}a_{22} + a_{22}^2$ = 不一定成立

(E) $A \pm B = \begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} \end{bmatrix}$ ，$tr(A\pm B) = a_{11} \pm b_{11} + a_{22} \pm b_{22}$

$= (a_{11}+a_{22}) \pm (b_{11}+b_{22})$

$= tr(A) \pm tr(B)$

2.

(A) $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$

(C) $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \dfrac{1}{\det(B)}\det(B)\det(A) = \det(A)$

(D) $\det(A^k) = \det(A \cdot A \cdot A \cdots A) = \det(A)\cdot\det(A)\cdot\det(A)\cdots\det(A) = \det(A)^k$

(E) e.g. $A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det(A+B) = 0 \neq \det(A) + \det(B) = 1$

3.

(A) A^{-1} 為可逆，B^{-1} 為可逆，但可逆+可逆不保證可逆.

(B) 保正半定而已

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 為 orthogonal set，但不為 LI

(D) 因左下角為 0，eigenvalue = T_{11} 之 eigenvalue $\cup$ T_{22} 之 eigenvalue

(E) ker(T) 必是 subspace

4.

(A) 沒有0

(B) 沒有0

(C) 不滿足加法封閉性

(E) 沒有0

5. N：$n\times n$ 佈於 $\mathbb{R}$ or $\mathbb{C}$，$N^k=0$

$f(x)=1-x^k=(1-x)(1+x+x^2+\cdots-x^{k-1})$

x 代入 $(-N)$，$=(I+N)\underbrace{(I-N+N^2+\cdots(-N)^{k-1})}_{=(I+N)^{-1}}=I$ (因為 $N^k=0$)

反矩陣存在

6. 單位橢圓 $=\left\{\begin{bmatrix}a\\b\end{bmatrix} \middle| \frac{1}{4}a^2+\frac{1}{9}b^2=1\right\}$

7. $\begin{bmatrix}-4-x & -5\\ 10 & 11-x\end{bmatrix}=(x-6)(x-1)$ $\lambda=6,1$ $\lambda(6)=\mathrm{span}\left\{\begin{bmatrix}-1/2\\1\end{bmatrix}\right\}$ $\lambda(1)=\mathrm{span}\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

$A=\begin{bmatrix}-1/2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}(1+6)^{100} & \\ & (1+1)^{100}\end{bmatrix}\begin{bmatrix}-1/2 & -1\\ 1 & 1\end{bmatrix}^{-1}$ 取 eigenvector 為 $\left\{\begin{bmatrix}-1/2\\1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\right\}$

8.
$$\begin{bmatrix}1+x_1 & x_2 & x_3 & \cdots & x_n\\ x_1 & 1+x_2 & x_3 & \cdots & x_n\\ \vdots & \vdots & 1+x_3 & \cdots & \vdots\\ x_1 & x_2 & x_3 & \cdots & 1+x_n\end{bmatrix}=\begin{bmatrix}1+x_1 & x_2 & x_3 & \cdots & x_n\\ -1 & 1 & 0 & \cdots & 0\\ -1 & 0 & 1 & \ddots & \vdots\\ \vdots & \vdots & \vdots & & \\ -1 & 0 & 0 & \cdots & 1\end{bmatrix}$$

$$=(1+x_1)\times\begin{bmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}_{(n-1)\times(n-1)}+x_2\times\begin{bmatrix}-1 & 1 & 0 & 0 & \\ -1 & 0 & 1 & 0 & \\ -1 & 0 & 0 & 1 & \ddots\\ \vdots & \vdots & \vdots & \vdots & \\ -1 & 0 & 0 & 0 & 1\end{bmatrix}_{(n-1)(n-1)}+\cdots\cdots$$

$$=1+x_1+x_2+x_3+\cdots+x_n=1+\sum_{i=1}^{n}x_i$$

9.

$x_1+x_2+\cdots+x_n<r$

$x_1+x_2+x_3+x_4<8$

$=x_1+x_2+x_3+x_4+r\le 7$

$\binom{7+4}{7}=\underline{330}$ #

10.

$\begin{pmatrix}1&2&3&4&5&6&7&8\\2&6&5&8&3&1&7&4\end{pmatrix}$

① $1\leftrightarrow 2\leftrightarrow 6$

② $3\leftrightarrow 5$

③ $4\leftrightarrow 8$

④ 7

$(1\,2\,6)\circ(3\,5)\circ(4\,8)\circ(7)$

4個等價類排列 $\underline{4!=24}$ #

11.

$\begin{cases}a_n=2a_{n-1}+n\\a_0=4\end{cases}$

$a_n^{(h)}=c_1\cdot 2^n$

$a_n^{(p)}=d_0+nd_1$

$d_0+nd_1-2(d_0+(n-1)d_1)=n$

代 $n=1$, $d_0+d_1-2d_0=1 \Rightarrow -d_0+d_1=1$

$n=2$, $d_0+2d_1-2(d_0+d_1) \Rightarrow -d_0=2, d_0=-2, d_1=-1$

$a_n=c_1\cdot 2^n-2-n$

代入 $a_0=c_1\cdot 2=4$

$c_1=6$

$\underline{a_n=6\cdot 2^n-2-n}$ #

12.

$a_n=6\cdot 2^n-2-n$, $n\ge 0$, 令 $A(x)$為 a_n 的生成函數，則 $A(x)=\sum_{n=0}^{\infty}a_nx^n$

$$\Rightarrow A(x)=\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}(6\cdot 2^n-2-n)x^n=6\sum_{n=0}^{\infty}(2x)^n-2\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}nx^n$$

$$=\frac{6}{1-2x}-\frac{2}{1-x}-B(x)$$，其中 $B(x)=\sum_{n=0}^{\infty}nx^n$

欲求 $B(x)$的 partial fraction decomposition

$$\sum_{i=0} \quad \sum_{i=0} \quad \sum_{i=0} 2x$$

$$= \frac{6}{1-2x} - \frac{x}{(1-x)^2} - \frac{2}{1-x}$$

$$= \frac{6}{1-2x} - \frac{x}{(1-x)^2} - \frac{2-2x}{(1-x)^2} = \frac{6}{1-2x} - \frac{-2+x}{(1-x)^2}$$

$$\rightarrow = \frac{a}{(1-x)^2} + \frac{b}{(1-x)} \Rightarrow \frac{-1}{(1-x)^2} + \frac{-1}{(1-x)}$$

$$= \frac{6}{1-2x} + \frac{(-1)}{(1-x)^2} + \frac{(-1)}{(1-x)} \#$$

13. 討論1出現次數之GF $= (1+x+x^2+\cdots)$

2出現次數之GF $= (1+x^2+x^4+\cdots)$

$\vdots$

n 〃 $= (1+x^n+x^{2n}+\cdots)$

則 $F(x) = \dfrac{1}{(1-x)(1-x^2)\cdots(1-x^n)}$

14. self-dual = 類似取not

self-dual function 所以 $(1,0,0) = (0,1,1)$

①總共有 $\xleftarrow{\quad m \quad}$ $2^m/2$ 種 input.

②又每一種 input 會對每個 element 討論取/不取

$$2^{2^{m-1}} \#$$

15. $(n-i)$: 外部節點 $= i+1$

① $n-1 = 2i$

② $\dfrac{n-1}{2} = i \#$