National Chiao Tung University, Department of Computer Science
DCP 3351 Probability Theory–Final Exam

Instructor: Dr. Wen-Hsiao Peng
Date: Wednesday, June 23, 2010
Time: 10:10pm – 12:30pm (Extra time may be granted upon request)
Format: Open book
Instructions:
1) You may give your answers in Chinese or English.
2) Please give your answers in succinct phrases or point form.
3) Please write your answers clearly (with explicit denotation of labels and symbols used).

1. (30 pts) True or False (Please reason)

   (a) Events A and B are independent if they are disjoint in sample space.

   (b) MAP estimator can always minimize error probability.

   (c) The CDF of $S_n = X_1 + X_2 + ... + X_n$ tends to normal as $n \to \infty$, where X_i are independent random variables with finite mean and variance.

   (d) If a random sequence Y_n converges in probability to c, then it also implies $\lim_{n\to\infty} E[Y_n] = c$.

   (e) If X, Y, Z are independent random variables, then X, Y are independent when conditioned on Z.

   (f) $E[aX^2 + b] = a\,(E[X])^2 + b$.

2. (20 pts) Given $f_{Y|X} \sim N(\rho x, \sigma^2)$ and that X distributes uniformly in $[-1, 1]$, compute the mean and the variance of Y. [Hint: $E[Y] = E_X E_{Y|X}[Y|X]$]

3. (50 pts) Suppose X is a random variable with Binomial distribution $B(n, p)$.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0, 1, ..., n.$$

   (a) Compute the maximum likelihood estimator $\widehat{p}_{ML}$ for the parameter p, given N independent observations $x_1, x_2, ..., x_N$.

   (b) Is $\widehat{p}_{ML}$ unbiased? Justify your answer.

   (c) Is $\widehat{p}_{ML}$ consistent? Justify your answer. [Hint: Weak Law of Large Number]

   (d) Find another estimator of the form $a\widehat{p}_{ML}$ that minimizes $E[(a\widehat{p}_{ML} - p)^2]$. Is the estimator biased, given that N is finite? True or False: unbiased estimators always lead to minimum mean square estimation error.

$$\text{Hint: } E[(\widehat{p} - p)^2] = var(\widehat{p}) + (E[\widehat{p}] - p)^2 = \text{Estimator Variance} + \text{Bias}^2$$

   (e) Compute the confidence interval $1 - \alpha = 0.6826$ with $N = 16$. [Hint: Use $p(1-p) \le \frac{1}{2}$ to approximate the variance]

4. (30 pts) Assume $Y = X^2 + W$ with $W \sim N(0, 1)$ being independent of X, which distributes uniformly in $[-1, 1]$.

   (a) Compute the covariance of X and Y. Are they correlated? [Hint: $E[XY] - \mu_x \mu_Y$]

   (b) Compute the MMSE estimator $\widehat{Y}_{MMSE}$ based on observing X.

   (c) Compute the LMMSE estimator $\widehat{Y}_{LMMSE}$ of the form $aX + b$.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Figure 1:

Enjoy Your Summer!