Probability: Homework Set Three, March 12, 2026
Due: March 27, 2026

1) The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}, \ n \geq k.$$

Give a combinatorial argument (no computations are needed) to establish this identity. Hint: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest numbered member?

2) The 'ménages' problem poses the following question. Suppose that men and women sit alternatively at a circular table. If n couples are seated randomly according to this rule, show that the probability that nobody sits next to his or her partner is

$$\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.$$

You may find it useful to show first that the number of ways of selecting k non-overlapping pairs of adjacent seats is $\binom{2n-k}{k} 2n(2n-k)^{-1}$.

3) From a group of 9 women and 7 men, a committee consisting of 4 men and 4 women is to be formed. How many different committees are possible if
   a) 2 of the men refuse to serve together;
   b) 2 of the women refuse to serve together;
   c) 1 man and 1 woman refuse to serve together?

4) A circular target of unit radius is divided into four annular zones with outer radii 1/4, 1/2, 3/4, and 1, respectively. Suppose 10 shots are fired independently and at random into the target.
   a) Compute the probability that at most three shots land in the zone bounded by the circles of radius 1/2 and radius 1.
   b) If 5 shots land inside the disk of radius 1/2, find the probability that at least one is in the disk of radius 1/4.

5) Andy, Betty, Chloë, and Dawn were all friends at school. Subsequently each of the $\binom{4}{2} = 6$ subpairs meet up; at each of the six meetings the pair involved quarrel with some fixed probability p, or become firm friends with probability $1 - p$. Quarrels take place independently of each other. In future, if any of the four hears a rumor, then he/she tells it to his/her firm friends only. If Andy hears a rumor, what is the probability that
   a) Dawn hears it?
   b) Dawn hears it if Andy and Betty have quarreled?
   c) Dawn hears it if Betty and Chloë have quarreled?
   d) Dawn hears it if Dawn has quarreled with Andy?

6) Problems 43, 44, and 45 from Chapter 1 of the textbook. **Note that these questions are for practice, review, or optional study purposes and do not need to be submitted.**