515511 Fall 2025: Probability

Homework #1 Solution

1. (a)

$$\begin{aligned} P(A) &= P(A \cap \Omega) \\ &= P(A \cap (\bigcup_{i=1}^{n} S_i)) \\ &= P(\bigcup_{i=1}^{n} (A \cap S_i)), \quad \textit{by Axiom 3} \\ &= \sum_{i=1}^{n} P(A \cap S_i) \end{aligned}$$

(b)

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A \cap B) + P(A \cap B^c \cap C^c) + P(A \cap B^c \cap C) \\ &= P(A \cap B) + P(A \cap B^c \cap C^c) + P(A \cap C \cap B^c) \end{aligned}$$

$$\begin{bmatrix} P(A \cap C) = P(A \cap C \cap B) + P(A \cap C \cap B^c) \\ \Rightarrow P(A \cap C \cap B^c) = P(A \cap C) - P(A \cap C \cap B) \end{bmatrix}$$

$$= P(A \cap B) + P(A \cap B^c \cap C^c) + P(A \cap C) - P(A \cap C \cap B)$$

3. Let A_i be the event of the ball randomly chosen from jar i is white

For $i = 1$, $P(A_1) = \frac{n}{m+n}$ holds

Suppose $i = k$, $P(A_k) = \frac{n}{m+n}$ holds

For $i = k + 1$,

$$\begin{aligned} P(A_{k+1}) &= \frac{n+1}{m+n+1} P(A_k) + \frac{n}{m+n+1}(1 - P(A_k)) \\ &= \frac{n+1}{m+n+1}(\frac{n}{m+n}) + \frac{n}{m+n+1}(\frac{m}{m+n}) \\ &= \frac{n(m+n+1)}{(m+n+1)(m+n)} \\ &= \frac{n}{m+n} \end{aligned}$$

By the principle of mathematical induction, $P(A_i) = \frac{n}{m+n}$ is true for $i \in \mathbb{N}$
Therefore, $P(A_1) = P(A_k) = \frac{n}{m+n}$

7.

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\begin{aligned}
P(\bigcup_{k=1}^{\infty} C_k) &= P(\bigcup_{k=1}^{\infty} R_k) \\
&= \sum_{k=1}^{\infty} P(R_k) \quad \textit{by Axiom 3 (for all } k, R_k \textit{ are disjoint events)} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} P(R_k) \\
&= \lim_{n\to\infty} P(\bigcup_{k=1}^{n} R_k) \quad \textit{by Axiom 3} \\
&= \lim_{n\to\infty} P(C_n) \quad (\textit{since } \bigcap_{k=1}^{n} R_n = C_n)
\end{aligned}
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