1. (a) $P(x) = \begin{cases} \frac{x^2}{a}, & x = \{-3,-2,-1,0,1,2,3\} \\ 0, & \text{otherwise} \end{cases}$

$\sum P(x) = 1 = \frac{1}{a}\sum_{x=-3}^{3} x^2 = \frac{28}{a} \Rightarrow a = 28$

$E[X] = \sum x P(x) = \frac{1}{a}\sum_{x=-3}^{3} x^3 = \frac{0}{a} = 0$

(b) $Z = (X - E[X])^2 = (X-0)^2 = X^2$

Pmf of $Z = X^2 \Rightarrow P(0) = 0$

$P(Z=1^2) = \frac{1}{28}$

$P(Z=2^2) = \frac{4}{28}$

$P(Z=3^2) = \frac{9}{28}$

(c) $Var(X) = E((X-E[X])^2) = E(Z) = 1\cdot\frac{1}{28}\cdot 2 + 4\cdot\frac{4}{28}\cdot 2 + 9\cdot\frac{9}{28}\cdot 2 = \frac{98}{14} = 7$

(d) $Var(X) = \sum_x (x - E[X])^2 \cdot P_x(x)$

$= \sum_x (x-0)^2 \cdot \frac{x^2}{28}$

$= \sum_{x=-3}^{3} \frac{x^4}{28} = \frac{196}{28} = 7$

2.

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

$P(x=0) = \frac{6}{36}$

$P(1) = \frac{10}{36}$

$P(2) = \frac{8}{36}$

$P(3) = \frac{6}{36}$

$P(4) = \frac{4}{36}$

$P(5) = \frac{2}{36}$

3. (a) $P(x) = \left(\frac{5}{6}\right)^{x-1}\cdot\frac{1}{6}$

(b) $\sum_{x=1}^{\infty} P(x) = \frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{5}{6}\right)^k = \frac{1}{6}\cdot\frac{1-\left(\frac{5}{6}\right)^{\infty}}{1-\frac{5}{6}} = 1$

(c) $P(x=1,3,5,7,\ldots) = \sum_{k=0}^{\infty}\frac{1}{6}\cdot\left(\frac{5}{6}\right)^{2k-1+1} = \frac{1}{6}\cdot\frac{1}{1-\frac{11}{36}} = \frac{6}{11}$

(d) $f(x) = P(X \le x) = 1 - P(X-x) = 1-(1-p)^x = 1-\left(\frac{5}{6}\right)^x$

4. (a) (i) $E[X] = \frac{1+2+3}{3} = 2$

$E[X^2] = \frac{1+4+9}{3} = \frac{14}{3}$

$V(X) = E[X^2] - (E[X])^2 = \frac{14}{3} - 4 = \frac{2}{3}$

(ii) $P(Y = 2X+1) = \frac{1}{3}$ for $Y \in \{3,5,7\}$

$E[Y] = E[2X+1] = 2E[X]+1 = 5$

$V(Y) = V(2X+1) = 2^2 V(X) = \frac{8}{3}$

(b) (i) $E[X] = \sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^x x = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$

$E[X^2] = \sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^x x^2 = \frac{\frac{1}{2}\cdot\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^3} = 6$

$V(X) = E[X^2] - (E[X])^2 = 6 - 4 = 2$

(ii) $P(Y=y) = \begin{cases} 2^{-x}, & \text{if } y = x^3 \text{ for } x \ge 1 \\ 0, & \text{otherwise} \end{cases}$

(c) (i) $E[X] = \frac{1+0-1}{3} = 0$

$E[X^2] = \frac{1+0+1}{3} = \frac{2}{3}$

$V(X) = E[X^2] - (E[X])^2 = \frac{2}{3}$

(ii) $Y = X^2$, $Y \in \{0,1\}$

$E[Y] = E[X^2] = \frac{2}{3}$

$E[Y^2] = \frac{1+0+1}{3} = \frac{2}{3}$

$V(Y) = E[Y^2] - (E[Y])^2 = \frac{2}{9}$

(d) $P(Y = x^4) = \begin{cases} \left(\frac{1}{2}\right)^{-x}, & \text{if } y = x^4 \text{ for some } x \in Z \\ 0, & \text{otherwise} \end{cases}$

5. $P_Z(z)$ is pmf if nonnegative and sum=1

$$P_Z(z) = \begin{cases} \frac{1}{4}, \text{ for } z=0 \\ \frac{5}{8}, \text{ for } z=1 \\ \frac{1}{4}\cdot\left(\frac{1}{2}\right)^{\sqrt{z}}, \text{ for } z=4,9,16,\dots \end{cases} \Rightarrow \text{nonnegative}$$

$$\sum P_Z(z) = \frac{1}{4}+\frac{5}{8}+\frac{1}{4}\cdot\sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^k = \frac{7}{8}+\frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} = 1 \Rightarrow \text{sum}=1$$

$\Rightarrow P_Z(z)$ is pmf

6. (a) $P(N=k) = C_k^{1000}\cdot(0.01)^k\cdot(0.99)^{1000-k}$

(b) $\lambda = n\cdot p = 1000\cdot 0.01 = 10$

$$P(N=k) \approx e^{-\lambda}\cdot\frac{\lambda^k}{k!} = e^{-10}\cdot\frac{10^k}{k!}, \ k=0,1,2,\dots$$

(c) $$P(N>50) = 1-\sum_{k=0}^{50} C_k^{1000}\cdot(0.01)^k\cdot(0.99)^{1000-k}$$

$$= 1-\sum_{k=0}^{50} e^{-10}\frac{10^k}{k!}$$