515511 Fall 2025: Probability

Homework #4 Solution

4. There are two hypotheses:

$$H_0 : \text{the phone number is } 2537267,$$
$$H_1 : \text{the phone number is not } 2537267,$$

and their prior probabilities are

$$\mathbf{P}(H_0) = \mathbf{P}(H_1) = 0.5.$$

Let B be the event that Artemisia obtains a busy signal when dialing this number. Under H_0, we expect a busy signal with certainty (since she is calling the number from the house phone itself):

$$\mathbf{P}(B|H_0) = 1.$$

Under H_1, the conditional probability of B (the probability of a random number being busy) is

$$\mathbf{P}(B|H_1) = 0.01.$$

Using Bayes' rule we obtain the posterior probability:

$$\begin{aligned}\mathbf{P}(H_0|B) &= \frac{\mathbf{P}(B|H_0)\mathbf{P}(H_0)}{\mathbf{P}(B|H_0)\mathbf{P}(H_0) + \mathbf{P}(B|H_1)\mathbf{P}(H_1)} \\ &= \frac{1 \times 0.5}{(1 \times 0.5) + (0.01 \times 0.5)} \\ &= \frac{0.5}{0.5 + 0.005} \\ &= \frac{0.5}{0.505} \\ &\approx 0.9901.\end{aligned}$$

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5. The waiting time X follows an exponential distribution with parameter Θ, so the likelihood function is $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}$. The prior PDF is given by:

$$f_\Theta(\theta) = \begin{cases} 10\theta, & \text{if } \theta \in [0, 1/5], \\ 0, & \text{otherwise.} \end{cases}$$

(a) Alvin waits $X = 30$ minutes. The likelihood is $f_{X|\Theta}(30|\theta) = \theta e^{-30\theta}$.
The posterior PDF is proportional to the product of the likelihood and the prior:

$$\begin{aligned}f_{\Theta|X}(\theta|30) &\propto f_{X|\Theta}(30|\theta) f_\Theta(\theta) \\ &\propto (\theta e^{-30\theta})(10\theta) \\ &\propto \theta^2 e^{-30\theta}, \quad \text{for } 0 \le \theta \le 1/5.\end{aligned}$$

To find the exact posterior PDF, we normalize it using constant c:

$$f_{\Theta|X}(\theta|30) = \frac{\theta^2 e^{-30\theta}}{c}, \quad \text{where } c = \int_0^{1/5} u^2 e^{-30u}\, du.$$

MAP Estimate: We maximize the unnormalized posterior $g(\theta) = \theta^2 e^{-30\theta}$.

$$\begin{aligned}\frac{d}{d\theta}(\theta^2 e^{-30\theta}) &= 2\theta e^{-30\theta} - 30\theta^2 e^{-30\theta} \\ &= \theta e^{-30\theta}(2 - 30\theta).\end{aligned}$$

Setting the derivative to 0, we get $2 - 30\theta = 0 \implies \theta = 1/15$. Since $1/15 \in [0, 1/5]$, this is the maximum.

$$\hat{\theta}_{\text{MAP}} = \frac{1}{15}.$$

Conditional Expectation Estimate (LMS):

$$\hat{\theta}_{\text{LMS}} = \mathbf{E}[\Theta|X = 30] = \int_0^{1/5} \theta f_{\Theta|X}(\theta|30)\, d\theta = \frac{\int_0^{1/5} \theta^3 e^{-30\theta}\, d\theta}{\int_0^{1/5} \theta^2 e^{-30\theta}\, d\theta}.$$

(b) Alvin records $\mathbf{x} = \{30, 25, 15, 40, 20\}$. Here $n = 5$ and $\sum x_i = 130$. The joint likelihood is $f_{\mathbf{X}|\Theta}(\mathbf{x}|\theta) = \theta^5 e^{-130\theta}$.

The posterior PDF is:

$$\begin{aligned}f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) &\propto (\theta^5 e^{-130\theta})(10\theta) \\ &\propto \theta^6 e^{-130\theta}, \quad \text{for } 0 \le \theta \le 1/5.\end{aligned}$$

Thus, $f_{\Theta|\mathbf{X}}(\theta|\mathbf{x}) = \frac{\theta^6 e^{-130\theta}}{k}$, where $k = \int_0^{1/5} u^6 e^{-130u}\, du$.

MAP Estimate: Maximize $h(\theta) = \theta^6 e^{-130\theta}$.

$$\frac{d}{d\theta}(\theta^6 e^{-130\theta}) = \theta^5 e^{-130\theta}(6 - 130\theta).$$

Setting derivative to 0: $6 - 130\theta = 0 \implies \theta = \frac{6}{130} = \frac{3}{65}$. Since $\frac{3}{65} \approx 0.046 < 0.2$, it is within the range.

$$\hat{\theta}_{\text{MAP}} = \frac{3}{65}.$$

Conditional Expectation Estimate (LMS):

$$\hat{\theta}_{\text{LMS}} = \mathbf{E}[\Theta|\mathbf{X} = \mathbf{x}] = \frac{\int_0^{1/5} \theta^7 e^{-130\theta}\, d\theta}{\int_0^{1/5} \theta^6 e^{-130\theta}\, d\theta}.$$

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6. From Bayes' rule,

$$\begin{aligned}
p_{\Theta|X}(\theta|x) &= \frac{p_{X|\Theta}(x|\theta)p_\Theta(\theta)}{p_X(x)} \\
&= \frac{p_{X|\Theta}(x|\theta)p_\Theta(\theta)}{\sum_{i=1}^{100} p_{X|\Theta}(x|i)p_\Theta(i)} \\
&= \frac{\frac{1}{\theta}\cdot\frac{1}{100}}{\sum_{i=x}^{100}\frac{1}{i}\cdot\frac{1}{100}} \\
&= \begin{cases} \frac{\frac{1}{\theta}}{\sum_{i=x}^{100}\frac{1}{i}}, & \text{for } \theta = x, x+1, \ldots, 100, \\ 0, & \text{for } \theta = 1, 2, \ldots, x-1. \end{cases}
\end{aligned}$$

Given that $X = x$, the posterior probability is maximized at $\hat{\theta} = x$, and this is the MAP estimate of Θ given x. The LMS estimate is

$$\hat{\theta} = \mathbf{E}[\Theta \mid X = x] = \sum_{\theta=1}^{100} \theta p_{\Theta|X}(\theta \mid x) = \frac{101 - x}{\sum_{i=x}^{100}\frac{1}{i}}.$$

Following figure plots the MAP and LMS estimates of Θ as a function of X.

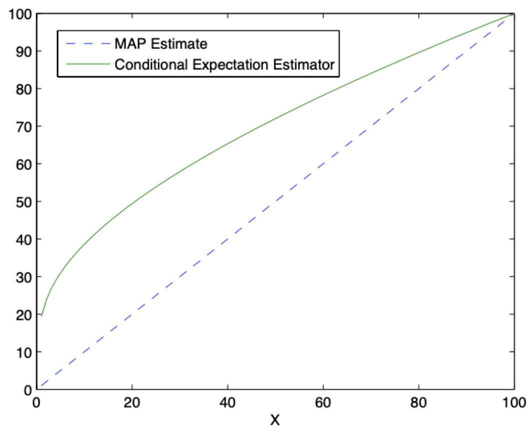

Figure 1: MAP and LMS estimates of Θ as a function of X