515511 Fall 2025: Probability

Homework #2 Solution

2.

$$\begin{aligned} q_n &= q_{n-1}(1-p) + (1-q_{n-1})p \\ &= q_{n-1}(1-p-p) + p \\ &= q_{n-1}(1-2p) + p \end{aligned}$$

$$\begin{array}{rrcl} & q_n & = & \cancel{q_{n-1}(1-2p)} + p \\ & \cancel{(1-2p)q_{n-1}} & = & \cancel{q_{n-2}(1-2p)^2} + p(1-2p) \\ & & \vdots & \\ +) & \cancel{(1-2p)^{n-2}q_2} & = & q_1(1-2p)^{n-1} + p(1-2p)^{n-2} \\ \hline & q_n & = & (1-p)(1-2p)^{n-1} + p(1+...+(1-2p)^{n-2}) \\ & & = & (1-2p)(1-2p)^{n-1} + p(1+...+(1-2p)^{n-2} + (1-2p)^{n-1}) \\ & & = & (1-2p)^n + p\frac{1-(1-2p)^n}{1-(1-2p)} \\ & & = & (1-2p)^n + \frac{1}{2}(1-(1-2p)^n) \\ & & = & \frac{1}{2}(1+(1-2p)^n) \end{array}$$

5.

$$\begin{aligned} & P(\text{the forth spade appears in the seventh draw}) \\ =& P(\text{three spades appear in first six draws})P(\text{spade in the seventh draw}) \\ =& \frac{C_3^{13}C_3^{39}}{C_6^{52}} \times \frac{10}{46} \approx 0.02791 \end{aligned}$$

6.

$$\begin{aligned} P(\text{system is up}) &= P(u_1 \cup (u_2 \cap u_3) \cup u_4) \\ &= 1 - P((u_1 \cup (u_2 \cap u_3) \cup u_4)^c) \\ &= 1 - P(u_1^c \cap (u_2 \cap u_3)^c \cap u_4^c) \end{aligned}$$

Since $u_1, ..., u_4$ are independent, $u_1^c, (u_2 \cap u_3)^c, u_4^c$ are independent, we have

$$\begin{aligned} P(\text{system is up}) &= 1 - P(u_1^c \cap (u_2 \cap u_3)^c \cap u_4^c) \\ &= 1 - P(u_1^c)P((u_1 \cap u_3)^c)P(u_4^c), \\ &= 1 - (1 - p_1)(1 - p_2 p_3)(1 - p_4) \end{aligned}$$

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Homework #3 Solution

1. (a)

$$\sum_x \frac{x^2}{a} = 1$$
$$\Rightarrow \frac{(-3)^2 + (-2)^2 + (-1)^2 + 0 + 1^2 + 2^2 + 3^2}{a} = 1$$
$$\Rightarrow a = 28$$
$$E[X] = (-3) \times \frac{9}{28} + (-2) \times \frac{4}{28} + ... + 2 \times \frac{4}{28} + 3 \times \frac{9}{28} = 0$$

(b)

$$Z = (X - E[X])^2 = (X - 0)^2 = X^2$$

$$P(z) = \begin{cases} \frac{z}{14}, & if \quad z = 0, 1, 4, 9 \\ 0, & otherwise. \end{cases}$$

(c)

$$Var(X) = E[X - E[X]]^2 = E[Z] = 1 \times \frac{1}{14} + 4 \times \frac{4}{14} + 9 \times \frac{9}{14} = 7$$

(d)

$$\begin{aligned} Var(X) &= \sum_x (x - E[X])^2 P(x) \\ &= \sum_x x^2 P(x) \\ &= (-3)^2 \times \frac{9}{28} + (-2)^2 \times \frac{4}{28} + ... + 2^2 \times \frac{4}{28} + 3^2 \times \frac{9}{28} \\ &= 7 \end{aligned}$$

2.

Difference	0	1	2	3	4	5
Count	6	10	8	6	4	2

$$P(x) = \begin{cases} \frac{6}{36}, & if \quad x = 0 \\ \frac{10}{36}, & if \quad x = 1 \\ \frac{8}{36}, & if \quad x = 2 \\ \frac{6}{36}, & if \quad x = 3 \\ \frac{4}{36}, & if \quad x = 4 \\ \frac{2}{36}, & if \quad x = 5 \\ 0, & otherwise. \end{cases}$$

3. (a)

$$P(x) = (\frac{1}{6})(\frac{5}{6})^{x-1}, \quad x = 1, 2, 3, ...$$

(b)

$$\sum_{x=1}^{\infty}(\frac{1}{6})(\frac{5}{6})^{x-1} = \frac{1}{6}\sum_{x=1}^{\infty}(\frac{5}{6})^{x-1} = \frac{1}{6} \times \frac{1}{1-\frac{5}{6}} = \frac{1}{6} \times 6 = 1$$

(c)

$$\begin{aligned} P(X = 1, 3, 5, 7, ...) &= \sum_{x=1,3,5,7,...}(\frac{1}{6})(\frac{5}{6})^{x-1} \\ &= \sum_{n=1}^{\infty}(\frac{1}{6})(\frac{5}{6})^{2n-2} \\ &= \frac{1}{6} \times \frac{1}{1-\frac{25}{36}} \\ &= \frac{1}{6} \times \frac{36}{11} \\ &= \frac{6}{11} \end{aligned}$$

(d)

$$P(X \leq x) = \sum_{n=1}^{x}(\frac{1}{6})(\frac{5}{6})^{n-1} = \frac{1}{6} \times \frac{(1-\frac{5}{6})^{x}}{1-\frac{5}{6}} = 1 - (\frac{5}{6})^{x}, \quad x = 1, 2, 3, ...$$

4. (a) (i)

$$E[X] = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = 2$$

$$Var(X) = E[x^2] = (E[X])^2 = (1 \times \frac{1}{3} + 2^2 \times \frac{1}{3} + 3^2 \times \frac{1}{3}) - 2^2 = \frac{14}{3} - 4 = \frac{2}{3}$$

(ii)

$$P_Y(y) = \begin{cases} \frac{1}{3}, & if \quad y = 3, 5, 7 \\ 0, & otherwise. \end{cases}$$

$$E[Y] = E[2X + 1] = 2E[X] + 1 = 5$$

$$Var(Y) = Var(2X + 1) = 2^2 Var(X) = 4 \times \frac{2}{3} = \frac{8}{3}$$

(b) (i)

$$E[X] = \sum_{x=1}^{\infty} x(\frac{1}{2})^2 = \frac{\frac{1}{2}}{(1 - \frac{1}{2})^2} = 2$$

Since

$$\begin{aligned} \sum_{n=0}^{\infty} a^n &= \frac{1}{1-a}, \quad where\ 0 < a < 1 \\ \Rightarrow \sum_{n=1}^{\infty} n a^{n-1} &= \frac{1}{(1-a)^2} \\ \Rightarrow \sum_{n=1}^{\infty} n a^n &= \frac{a}{(1-a)^2} \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \sum_{x=1}^{\infty} x^2 (\frac{1}{2})^x - 2^2 \\ &= \frac{\frac{1}{2} + \frac{1}{4}}{(\frac{1}{2})^3} - 2^2 = 2 \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} n a^n &= \frac{a}{(1-a)^2} \\ \Rightarrow \sum_{n=1}^{\infty} n^2 a^{n-1} &= \frac{1+a}{(1-a)^3} \\ \Rightarrow \sum_{n=1}^{\infty} n^2 a^n &= \frac{a+a^2}{(1-a)^3} \end{aligned}$$

(ii)

$$P_Y(y) = (\frac{1}{2})^{y^{\frac{1}{3}}}, \quad y = 1, 8, 27, 64, ...$$

(c) (i)

$$E[X] = \frac{1}{3} \times (-1) + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$$

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= (\frac{1}{3} \times (-1)^2 + \frac{1}{3} \times 0 + \frac{1}{3}1^2) - 0 \\ &= \frac{2}{3} \end{aligned}$$

(ii)

$$P_Y(y) = \begin{cases} \frac{2}{3}, & if \quad y = 1 \\ \frac{1}{3}, & if \quad y = 0 \\ 0, & otherwise. \end{cases}$$

$$E[Y] = \frac{2}{3} \times 1 + \frac{1}{3} \times 0 = \frac{2}{3}$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = (\frac{2}{3} \times 1^2 + \frac{1}{3} \times 0) - (\frac{2}{3})^2 = \frac{2}{9}$$

(d)

$$P_Y(y) = (\frac{1}{2})^{y^{\frac{1}{4}}}, \quad y = 1, 16, 81, 256, ...$$

5. (1) What to show $0 \leq P_Z(z) \leq 1$ for all z.

For $z = 4, 9, 16, ...$

$$\begin{aligned} &P_Z(z) = \frac{1}{4}(\frac{1}{2})^{\sqrt{z}} = (\frac{1}{2})^{\sqrt{z}+2} \geq 0 \\ &\because \sqrt{z} + 2 > 0 \quad z = 4, 9, 16, ... \\ &\therefore 0 \leq P_Z(z) \leq 1 \end{aligned}$$

For $z \neq 4, 9, 16, ...$, trivially, $0 \leq P_Z(z) \leq 1$.

(2) What to show: $\sum_Z P_Z(z) = 1$.

$$\begin{aligned}
\sum_z P_Z(z) &= \frac{1}{4} + \frac{5}{8} + \sum_{z=4,9,16,\dots} \frac{1}{4}(\frac{1}{2})^{\sqrt{z}} \\
&= \frac{1}{4} + \frac{5}{8} + \sum_{n=2}^{\infty} \frac{1}{4}(\frac{1}{2})^n \\
&= \frac{1}{4} + \frac{5}{8} + \frac{\frac{1}{16}}{1-\frac{1}{2}} \\
&= \frac{1}{4} + \frac{5}{8} + \frac{1}{8} \\
&= 1
\end{aligned}$$

By (1) and (2), $P_Z(z)$ is a pmf.

6. (a) Let X be the number of modems in use at the given time

$$P_X(x) = \begin{cases} \binom{1000}{x}(0.01)^x(0.99)^{1000-x}, & if \quad 0 \le x < 50 \\ \sum_{n=50}^{1000}\binom{1000}{n}(0.01)^n(0.99)^{1000-n}, & if \quad x = 50 \\ 0, & otherwise. \end{cases}$$

(b) Let Y be the number of customers that need a connection

$$P_Y(y) = e^{-10} \times \frac{10^k}{k!}$$

(c)

$$\begin{aligned}
exact : P(Y > 50) &= \sum_{k=51}^{1000} \binom{1000}{k}(0.01)^k(0.99)^{1000-k} \\
approximate : P(Y > 50) &= \sum_{k=51}^{1000} e^{-10} \times \frac{10^k}{k!}
\end{aligned}$$