HW1 Solution

Problem 1

Let $D = A \cup B$. Using the given identity $\mathbf{P}(X \cup Y) = \mathbf{P}(X) + \mathbf{P}(X^c \cap Y)$, we have:

$$\begin{aligned}
\mathbf{P}(A \cup B \cup C) &= \mathbf{P}(D \cup C) \\
&= \mathbf{P}(D) + \mathbf{P}(D^c \cap C) \\
&= \mathbf{P}(A \cup B) + \mathbf{P}((A \cup B)^c \cap C)
\end{aligned}$$

Applying the given identity to $\mathbf{P}(A \cup B)$ and De Morgan's Law to $(A \cup B)^c$:

$$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$$

Problem 2

Let the sample space be $S = \{s_1, s_2, s_3, \dots\}$.

a)

Assume all points are equally likely, i.e., $\mathbf{P}(s_i) = c \geq 0$ for all $i \in \mathbb{N}$. By the axioms of probability, $\sum_{i=1}^{\infty} \mathbf{P}(s_i) = 1$.

- If $c = 0$, then $\sum_{i=1}^{\infty} 0 = 0 \neq 1$.
- If $c > 0$, then $\sum_{i=1}^{\infty} c = \infty \neq 1$.

Both cases yield a contradiction. Thus, not all points can be equally likely.

b)

Yes.
Example: Let $\mathbf{P}(s_i) = \left(\frac{1}{2}\right)^i$ for $i = 1, 2, 3, \dots$
Clearly, $\mathbf{P}(s_i) > 0$ for all i. Furthermore, the sum forms a valid probability space:

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1/2}{1 - 1/2} = 1$$

Problem 3

The tournament has 2^n competitors. The total number of matches is:

$$\sum_{k=1}^{n} 2^{n-k} = 2^{n-1} + 2^{n-2} + \cdots + 1 = 2^n - 1$$

Since the initial draw is specified, the tournament outcome is fully determined by the winner of each match. Each match has 2 possible outcomes.
The sample space Ω is the set of all possible outcome sequences for the $2^n - 1$ matches:

$$\Omega = \{0,1\}^{2^n-1}, |\Omega| = 2^{2^n-1}$$

where 1 denotes the first player winning and 0 denotes the second player winning for each respective match.

Problem 4

Let H denote Heads and T denote Tails.

a)

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

b)

$$\Omega = \{H, TH, TTH, TTTH, TTTTH, \dots\} = \{T^k H \mid k \geq 0\}$$

Problem 5

Given $\mathbf{P}(A) = \frac{3}{4}$ and $\mathbf{P}(B) = \frac{1}{3}$.

Bounds for $\mathbf{P}(A \cap B)$

Upper bound: Since $(A \cap B) \subseteq A, B$, $\mathbf{P}(A \cap B) \leq \min(\mathbf{P}(A), \mathbf{P}(B)) = \frac{1}{3}$.
Lower bound: Since $\mathbf{P}(A \cup B) \leq 1$, we have:

$$\mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) \leq 1 \implies \frac{3}{4} + \frac{1}{3} - 1 \leq \mathbf{P}(A \cap B) \implies \frac{1}{12} \leq \mathbf{P}(A \cap B)$$

Thus, $\frac{1}{12} \leq \mathbf{P}(A \cap B) \leq \frac{1}{3}$.

Examples for Extremes

Upper extreme ($\frac{1}{3}$): Occurs if $B \subset A$. Then $A \cap B = B$, yielding $\mathbf{P}(A \cap B) = \frac{1}{3}$.
Lower extreme ($\frac{1}{12}$): Occurs if $A \cup B = S$. Then $\mathbf{P}(A \cup B) = 1$, yielding $\mathbf{P}(A \cap B) = \frac{3}{4} + \frac{1}{3} - 1 = \frac{1}{12}$.

Bounds for $\mathbf{P}(A \cup B)$

Using $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \frac{13}{12} - \mathbf{P}(A \cap B)$:

$$\frac{13}{12} - \frac{1}{3} \leq \mathbf{P}(A \cup B) \leq \frac{13}{12} - \frac{1}{12} \implies \frac{3}{4} \leq \mathbf{P}(A \cup B) \leq 1$$

Problem 6

Let C and D be the events of loving comics and chocolate, respectively.
Given: $\mathbf{P}(C) = 0.6$, $\mathbf{P}(D) = 0.7$, and $\mathbf{P}(C \cap D) = 0.4$.
We need to find $\mathbf{P}(C^c \cap D^c)$. By De Morgan's Law:

$$\mathbf{P}(C^c \cap D^c) = \mathbf{P}((C \cup D)^c) = 1 - \mathbf{P}(C \cup D)$$

Calculate $\mathbf{P}(C \cup D)$:

$$\mathbf{P}(C \cup D) = \mathbf{P}(C) + \mathbf{P}(D) - \mathbf{P}(C \cap D) = 0.6 + 0.7 - 0.4 = 0.9$$

Therefore:

$$\mathbf{P}(C^c \cap D^c) = 1 - 0.9 = 0.1$$