Probability: Homework Set Two

Reference Solutions

Problem 1

(a)

Proof by Mathematical Induction:
Base Case: For $n = 1$, $P(A_1) \leq P(A_1)$ holds trivially. For $n = 2$, by the addition rule of probability:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$$

since $P(A_1 \cap A_2) \geq 0$.
Assume the inequality holds for $n = k$, i.e., $P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i)$.
For $n = k + 1$:

$$\begin{aligned} P\left(\bigcup_{i=1}^{k+1} A_i\right) &= P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right) \\ &\leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) \quad \text{(using base case } n = 2\text{)} \\ &\leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1}) \quad \text{(using inductive hypothesis)} \\ &= \sum_{i=1}^{k+1} P(A_i) \end{aligned}$$

By mathematical induction, the inequality holds for all $n \geq 1$.

(b)

Proof:
Using De Morgan's Laws, $(\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c$.
We know that $P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$.
By Boole's inequality from part (a), applied to events A_i^c:

$$P\left(\bigcup_{i=1}^{n} A_i^c\right) \leq \sum_{i=1}^{n} P(A_i^c)$$

Multiplying by -1 reverses the inequality:

$$-P\left(\bigcup_{i=1}^{n} A_i^c\right) \geq -\sum_{i=1}^{n} P(A_i^c)$$

Adding 1 to both sides yields:

$$1 - P\left(\bigcup_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

Therefore,

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$

Problem 2

Let R be the event "it rains tomorrow", and L be the event "Joe is late".
Given: $P(R) = 0.7$, $P(R^c) = 0.3$, $P(L|R) = 0.3$, $P(L|R^c) = 0.1$.
We deduce $P(L^c|R) = 1 - 0.3 = 0.7$ and $P(L^c|R^c) = 1 - 0.1 = 0.9$.

(a)

By the Law of Total Probability:

$$\begin{aligned} P(L^c) &= P(L^c|R)P(R) + P(L^c|R^c)P(R^c) \\ &= (0.7)(0.7) + (0.9)(0.3) \\ &= 0.49 + 0.27 = 0.76 \end{aligned}$$

(b)

By Bayes' Theorem:

$$\begin{aligned} P(R|L^c) &= \frac{P(L^c|R)P(R)}{P(L^c)} \\ &= \frac{0.49}{0.76} = \frac{49}{76} \approx 0.6447 \end{aligned}$$

Problem 3

Given $P(N = i) = 2^{-i}$ for $i \geq 1$. Let S be the sum of scores.

(a)

Using Bayes' Theorem:

$$P(N = 2|S = 4) = \frac{P(S = 4|N = 2)P(N = 2)}{P(S = 4)}$$

To find $P(S=4)$, we sum over all possible N. The sum can only be 4 if $1 \leq N \leq 4$.
For $N=1$: $P(S=4|N=1)=\frac{1}{6}$.
For $N=2$: Possible outcomes to get sum 4 are $(1,3),(2,2),(3,1)$. $P(S=4|N=2)=\frac{3}{36}=\frac{1}{12}$.
For $N=3$: Possible outcomes to get sum 4 are $(1,1,2),(1,2,1),(2,1,1)$. $P(S=4|N=3)=\frac{3}{216}=\frac{1}{72}$.
For $N=4$: The only outcome is $(1,1,1,1)$. $P(S=4|N=4)=\frac{1}{1296}$.
Now compute $P(S=4)$:

$$\begin{aligned}
P(S=4) &= \sum_{i=1}^{4} P(S=4|N=i)P(N=i) \\
&= \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{12}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{72}\right)\left(\frac{1}{8}\right)+\left(\frac{1}{1296}\right)\left(\frac{1}{16}\right) \\
&= \frac{1}{12}+\frac{1}{48}+\frac{1}{576}+\frac{1}{20736} \\
&= \frac{1728+432+36+1}{20736}=\frac{2197}{20736}
\end{aligned}$$

Returning to Bayes' Theorem:

$$P(N=2|S=4)=\frac{\frac{1}{48}}{\frac{2197}{20736}}=\frac{\frac{432}{20736}}{\frac{2197}{20736}}=\frac{432}{2197}$$

(b)

Find $P(S=4|\text{N is even})$.

$$P(\text{N is even})=\sum_{k=1}^{\infty}P(N=2k)=\sum_{k=1}^{\infty}2^{-2k}=\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k}=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$$

For $S=4$ and N is even, N can only be 2 or 4.

$$\begin{aligned}
P(S=4\cap\text{N is even}) &= P(S=4\cap N=2)+P(S=4\cap N=4) \\
&= P(S=4|N=2)P(N=2)+P(S=4|N=4)P(N=4) \\
&= \frac{1}{48}+\frac{1}{20736}=\frac{432}{20736}+\frac{1}{20736}=\frac{433}{20736}
\end{aligned}$$

Thus,

$$P(S=4|\text{N is even})=\frac{P(S=4\cap\text{N is even})}{P(\text{N is even})}=\frac{\frac{433}{20736}}{\frac{1}{3}}=\frac{433}{6912}$$

Problem 4

(a)

The initial urn has r red and b black balls.

$$P(R_1)=\frac{r}{r+b}$$

(b)

By the Law of Total Probability:

$$\begin{aligned} P(R_2) &= P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1) \\ &= \left(\frac{r+c}{r+b+c}\right)\left(\frac{r}{r+b}\right) + \left(\frac{r}{r+b+c}\right)\left(\frac{b}{r+b}\right) \\ &= \frac{r(r+c)+rb}{(r+b+c)(r+b)} = \frac{r(r+b+c)}{(r+b+c)(r+b)} = \frac{r}{r+b} \end{aligned}$$

(c)

Guess: $P(R_j) = \frac{r}{r+b}$ for all $j \geq 1$.
Proof: Let X_i be the indicator variable for drawing a red ball on the i-th draw, meaning $P(R_i) = E[X_i]$. Before the $(n+1)$-th draw, exactly n draws have occurred. The total number of balls is deterministically $r+b+nc$, and the number of red balls is $r + c\sum_{i=1}^{n} X_i$.

Base Case ($n = 1$):

$$P(R_1) = E[X_1] = \frac{r}{r+b}$$

Inductive Step: Assume $P(R_i) = E[X_i] = \frac{r}{r+b}$ for all integers $1 \leq i \leq n$. For the $(n+1)$-th draw, the conditional probability of drawing red, given all previous draws, is:

$$E[X_{n+1} \mid X_1, \ldots, X_n] = \frac{r + c\sum_{i=1}^{n} X_i}{r+b+nc}$$

Taking the unconditional expectation of both sides (which allows us to pass the expectation inside the sum) and applying the inductive hypothesis $E[X_i] = \frac{r}{r+b}$:

$$\begin{aligned} E[X_{n+1}] &= \frac{r + c\sum_{i=1}^{n} E[X_i]}{r+b+nc} \\ &= \frac{r + cn\left(\frac{r}{r+b}\right)}{r+b+nc} \\ &= \frac{\frac{r(r+b)+cnr}{r+b}}{r+b+nc} \\ &= \frac{r(r+b+nc)}{(r+b)(r+b+nc)} \\ &= \frac{r}{r+b} \end{aligned}$$

By mathematical induction, $P(R_j) = E[X_j] = \frac{r}{r+b}$ holds for all $j \geq 1$.

(d)

Using the definition of conditional probability:

$$\begin{aligned} P(R_1|R_2) &= \frac{P(R_1 \cap R_2)}{P(R_2)} \\ &= \frac{P(R_2|R_1)P(R_1)}{P(R_2)} \\ &= \frac{\left(\frac{r+c}{r+b+c}\right)\left(\frac{r}{r+b}\right)}{\frac{r}{r+b}} \\ &= \frac{r+c}{r+b+c} \end{aligned}$$

Problem 5

The reasoning is flawed because it confuses probability with physical reality.

Suppose that each initial state (BB, BW, WB, WW) has a probability $\frac{1}{4}$ is just a mathematical guess. The calculated $\frac{2}{3}$ probability is simply the average expectation of drawing a black ball based on those initial guesses. However, an expected probability of $\frac{2}{3}$ does not mean that the urn physically contains exactly two black balls and one white ball. The actual contents of the urn are still unknown.

Problem 6

Let A be the reasoning prisoner, and B, C be the others. Let R_i be the event that prisoner i is released. The prior probabilities are $P(R_A) = P(R_B) = P(R_C) = \frac{2}{3}$. Exactly two are released, so the possible pairs released are $\{A, B\}, \{A, C\}, \{B, C\}$, each with probability $\frac{1}{3}$.

Let G_B be the event that the guard names B as a released prisoner. We find $P(R_A|G_B)$:

$$P(R_A|G_B) = \frac{P(G_B|R_A)P(R_A)}{P(G_B)}$$

If A is released, the actual released pair is either $\{A, B\}$ or $\{A, C\}$ with equal probability. If $\{A, B\}$ is released, the guard must name B (since the guard cannot name A). So, $P(G_B|\{A, B\}) = 1$. If $\{A, C\}$ is released, the guard must name C. So, $P(G_B|\{A, C\}) = 0$. Thus, $P(G_B|R_A) = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$.

To find $P(G_B)$, we apply the Law of Total Probability over the pairs:

$$\begin{aligned} P(G_B) &= P(G_B|\{A, B\})P(\{A, B\}) + P(G_B|\{A, C\})P(\{A, C\}) + P(G_B|\{B, C\})P(\{B, C\}) \\ &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

Returning to Bayes' theorem:

$$P(R_A|G_B) = \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2}} = \frac{2}{3}$$

The prisoner's flaw is assuming that $P(G_B|R_A) = 1$, ignoring the fact that if A is indeed released, there is a 50% chance the guard would have named C instead of B. The new information perfectly cancels out, leaving A's probability of release unchanged at $\frac{2}{3}$.