Probability: Homework Set Four

Reference Solutions

Problem 1

For a function to be a valid probability mass function (PMF) on the positive integers $x \in \{1, 2, \dots\}$, it must satisfy $\sum_{x=1}^{\infty} f(x) = 1$. We use this condition to find the constant C.

a) Geometric: $f(x) = C2^{-x}$

$$\sum_{x=1}^{\infty} C2^{-x} = C\left(\frac{1/2}{1 - 1/2}\right) = C(1) = 1$$
$$\implies C = 1$$

b) Logarithmic: $f(x) = C2^{-x}/x$ Using the Taylor series expansion for $\ln(1-y) = -\sum_{x=1}^{\infty} \frac{y^x}{x}$ for $|y| < 1$, we set $y = 1/2$:

$$\sum_{x=1}^{\infty} C\frac{(1/2)^x}{x} = C\left(-\ln\left(1 - \frac{1}{2}\right)\right) = C\ln 2 = 1$$
$$\implies C = \frac{1}{\ln 2}$$

c) Inverse square: $f(x) = Cx^{-2}$ Using the known sum of the Basel problem, $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$:

$$\sum_{x=1}^{\infty} C\frac{1}{x^2} = C\left(\frac{\pi^2}{6}\right) = 1$$
$$\implies C = \frac{6}{\pi^2}$$

d) 'Modified' Poisson: $f(x) = C2^x/x!$ Using the Taylor series for $e^y = \sum_{x=0}^{\infty} \frac{y^x}{x!}$, we can write $\sum_{x=1}^{\infty} \frac{y^x}{x!} = e^y - 1$. Setting $y = 2$:

$$\sum_{x=1}^{\infty} C\frac{2^x}{x!} = C(e^2 - 1) = 1$$
$$\implies C = \frac{1}{e^2 - 1}$$

Problem 2

1) Geometric: $f(x) = 2^{-x}$

- $P(X > 1) = 1 - P(X = 1) = 1 - \frac{1}{2} = \frac{1}{2}$.
- **Most probable value:** Since $f(x) = 2^{-x}$ is strictly decreasing for $x \geq 1$, the maximum occurs at $x = 1$.
- $P(X \text{ is even}) = \sum_{k=1}^{\infty} 2^{-2k} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1/4}{1-1/4} = \frac{1}{3}$.

2) Logarithmic: $f(x) = \frac{1}{\ln 2} \frac{2^{-x}}{x}$

- $P(X > 1) = 1 - P(X = 1) = 1 - \frac{1}{2 \ln 2}$.
- **Most probable value:** Both 2^{-x} and $1/x$ are strictly decreasing for $x \geq 1$, so $f(x)$ is strictly decreasing. The maximum occurs at $x = 1$.
- $P(X \text{ is even}) = \sum_{k=1}^{\infty} \frac{1}{\ln 2} \frac{2^{-2k}}{2k} = \frac{1}{2 \ln 2} \sum_{k=1}^{\infty} \frac{(1/4)^k}{k} = \frac{1}{2 \ln 2} \left(-\ln\left(1 - \frac{1}{4}\right)\right) = \frac{\ln(4/3)}{\ln 4}$.

3) Inverse square: $f(x) = \frac{6}{\pi^2 x^2}$

- $P(X > 1) = 1 - P(X = 1) = 1 - \frac{6}{\pi^2}$.
- **Most probable value:** Since $1/x^2$ is strictly decreasing for $x \geq 1$, the maximum occurs at $x = 1$.
- $P(X \text{ is even}) = \sum_{k=1}^{\infty} \frac{6}{\pi^2 (2k)^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{6}{\pi^2 k^2} = \frac{1}{4}(1) = \frac{1}{4}$.

4) Modified Poisson: $f(x) = \frac{1}{e^2-1} \frac{2^x}{x!}$

- $P(X > 1) = 1 - P(X = 1) = 1 - \frac{2}{e^2-1} = \frac{e^2-3}{e^2-1}$.
- **Most probable value:** Consider the ratio $\frac{f(x+1)}{f(x)} = \frac{2^{x+1}/(x+1)!}{2^x/x!} = \frac{2}{x+1}$. For $x = 1$, the ratio is 1, meaning $f(1) = f(2)$. For $x \geq 2$, the ratio is less than 1, meaning the PMF decreases. Thus, the most probable values are $x = 1$ and $x = 2$.
- $P(X \text{ is even}) = \sum_{k=1}^{\infty} \frac{2^{2k}}{(e^2-1)(2k)!}$. Using the hyperbolic cosine series $\cosh(y) = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!}$, we have $\sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} = \cosh(2) - 1$.

$$P(X \text{ is even}) = \frac{\cosh(2) - 1}{e^2 - 1} = \frac{\frac{e^2+e^{-2}}{2} - 1}{e^2 - 1} = \frac{e^4 - 2e^2 + 1}{2e^2(e^2 - 1)} = \frac{(e^2 - 1)^2}{2e^2(e^2 - 1)} = \frac{e^2 - 1}{2e^2} = \frac{1 - e^{-2}}{2}.$$

Problem 3

Is it generally true that $E[1/X] = 1/E[X]$?
No, it is not generally true. By Jensen's Inequality, since the function $g(x) = 1/x$ is strictly convex for $x > 0$, we have $E[1/X] > 1/E[X]$ for any non-degenerate random variable X that takes positive values.

Is it ever true that $E[1/X] = 1/E[X]$?
Yes, it is true if and only if X is a degenerate random variable (a constant) with probability 1. That is, $P(X = c) = 1$ for some constant $c \neq 0$. In this case, $E[1/X] = 1/c$ and $1/E[X] = 1/c$.

Problem 4

Let X be a non-negative integer-valued random variable. By definition of the expected value:

$$E[X] = \sum_{x=1}^{\infty} xP(X = x)$$

We can express the integer x as a sum of ones: $x = \sum_{n=0}^{x-1} 1$. Substituting this into the expectation formula gives:

$$E[X] = \sum_{x=1}^{\infty} \left(\sum_{n=0}^{x-1} 1 \right) P(X = x) = \sum_{x=1}^{\infty} \sum_{n=0}^{x-1} P(X = x)$$

Since all terms are non-negative, we can interchange the order of summation (Tonelli's Theorem). The region of summation is $0 \leq n < x < \infty$. Reordering the sums yields:

$$E[X] = \sum_{n=0}^{\infty} \sum_{x=n+1}^{\infty} P(X = x)$$

Recognizing that the inner sum represents the probability that X is strictly greater than n:

$$E[X] = \sum_{n=0}^{\infty} P(X > n)$$

Problem 5

Given the PMF: $P(X = k) = \binom{n+k-1}{k} p^n (1-p)^k$ for $k = 0, 1, \ldots$.

a) Show that $E[X] = \frac{n(1-p)}{p}$

$$E[X] = \sum_{k=0}^{\infty} k \binom{n+k-1}{k} p^n (1-p)^k = \sum_{k=1}^{\infty} k \frac{(n+k-1)!}{k!(n-1)!} p^n (1-p)^k$$
$$= \sum_{k=1}^{\infty} n \frac{(n+k-1)!}{(k-1)!n!} p^n (1-p)^k = \sum_{k=1}^{\infty} n \binom{n+k-1}{k-1} p^n (1-p)^k$$

Let $j = k - 1$. As k goes from 1 to ∞, j goes from 0 to ∞:

$$\begin{aligned} E[X] &= \sum_{j=0}^{\infty} n \binom{n+j}{j} p^n (1-p)^{j+1} \\ &= \frac{n(1-p)}{p} \sum_{j=0}^{\infty} \binom{(n+1)+j-1}{j} p^{n+1} (1-p)^j \end{aligned}$$

The sum represents the total probability of a Pascal distribution with parameters $(n + 1, p)$, which must sum to 1.

$$E[X] = \frac{n(1-p)}{p} \cdot 1 = \frac{n(1-p)}{p}$$

b) Show that $\sigma_X^2 = \frac{n(1-p)}{p^2}$ First, we find $E[X(X-1)]$:

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{\infty} k(k-1) \binom{n+k-1}{k} p^n (1-p)^k \\ &= \sum_{k=2}^{\infty} k(k-1) \frac{(n+k-1)!}{k(k-1)(k-2)!(n-1)!} p^n (1-p)^k \\ &= \sum_{k=2}^{\infty} n(n+1) \frac{(n+k-1)!}{(k-2)!(n+1)!} p^n (1-p)^k \\ &= n(n+1) \sum_{k=2}^{\infty} \binom{n+k-1}{k-2} p^n (1-p)^k \end{aligned}$$

Let $j = k - 2$:

$$\begin{aligned} E[X(X-1)] &= n(n+1) \sum_{j=0}^{\infty} \binom{n+j+1}{j} p^n (1-p)^{j+2} \\ &= \frac{n(n+1)(1-p)^2}{p^2} \sum_{j=0}^{\infty} \binom{(n+2)+j-1}{j} p^{n+2} (1-p)^j \end{aligned}$$

Again, the sum is over a valid PMF (parameters $n + 2, p$) and equals 1.

$$E[X(X-1)] = \frac{n(n+1)(1-p)^2}{p^2}$$

Now, calculate the variance:

$$\begin{aligned} \sigma_X^2 &= E[X(X-1)] + E[X] - (E[X])^2 \\ &= \frac{n(n+1)(1-p)^2}{p^2} + \frac{n(1-p)}{p} - \left(\frac{n(1-p)}{p}\right)^2 \\ &= \frac{n^2(1-p)^2 + n(1-p)^2 + np(1-p) - n^2(1-p)^2}{p^2} \\ &= \frac{n(1-p)^2 + np(1-p)}{p^2} = \frac{n(1-p)[(1-p)+p]}{p^2} = \frac{n(1-p)}{p^2} \end{aligned}$$