Probability: Homework Set Five

Reference Solutions

Problem 1

Given X, Y are independent and $P(X = x, Y = y) = \frac{1}{4}$ for $x, y \in \{1, -1\}$. The marginal distribution of Z is:

- $P(Z = 1) = P(X = 1, Y = 1) + P(X = -1, Y = -1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- $P(Z = -1) = P(X = 1, Y = -1) + P(X = -1, Y = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Pairwise Independence:
Check X and Z: $P(X = 1, Z = 1) = P(X = 1, Y = 1) = \frac{1}{4}$. Since $P(X = 1)P(Z = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, X and Z are independent. By symmetry, Y and Z are also independent. Combined with the given independence of X and Y, X, Y, Z are **pairwise independent**.
Mutual Independence:
Check $P(X = 1, Y = 1, Z = -1)$. Since $Z = XY$, if $X = 1, Y = 1$, then Z must be 1. Thus, $P(X = 1, Y = 1, Z = -1) = 0$. However, $P(X = 1)P(Y = 1)P(Z = -1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq 0$. Since the joint probability does not equal the product of the marginals, they are **not independent**.

Problem 2

Since each egg hatches with probability p independently, the number of chicks $K \sim \text{Binomial}(N, p)$. Given $N \sim \text{Poisson}(\lambda)$, and for a fixed N, $K \sim \text{Binomial}(N, p)$.

- **Conditional Expectation $E[K|N]$:**
  For a Binomial distribution, the mean is np. Thus, $E[K|N] = Np$.
- **Expectation $E[K]$:**
  Using the Law of Iterated Expectations:
  $$E[K] = E[E[K|N]] = E[Np] = pE[N]$$
  Since $N \sim \text{Poisson}(\lambda)$, $E[N] = \lambda$. Therefore, $E[K] = p\lambda$.

Problem 3

Mean and Variance:

- $E[Y] = E[\sum X_k] = \sum E[X_k] = \sum p_k$.

- Since X_k are independent, $var(Y) = \sum var(X_k) = \sum p_k(1-p_k)$.

Maximization:
Let $C = E[Y] = \sum p_k$. We want to maximize $var(Y) = \sum p_k - \sum p_k^2 = C - \sum p_k^2$. This is equivalent to minimizing $\sum p_k^2$ subject to $\sum p_k = C$. By the Cauchy-Schwarz inequality:

$$\left(\sum_{k=1}^{n} 1 \cdot p_k\right)^2 \leq \left(\sum_{k=1}^{n} 1^2\right)\left(\sum_{k=1}^{n} p_k^2\right) \implies C^2 \leq n \sum p_k^2 \implies \sum p_k^2 \geq \frac{C^2}{n}$$

Equality holds if and only if $p_1 = p_2 = \cdots = p_n = C/n$. Thus, the variance is maximized when all p_k are equal.

Problem 4

(a) $p_T(x) = (N+1)^{-1}$ **for** $x \in \{0, 1, \ldots, N\}$**:**
Given $T > t$, T is uniform on $\{t+1, \ldots, N\}$.

$$E[T|T > t] = \frac{(t+1) + N}{2}$$

Subsequent lifetime: $E[T - t|T > t] = \frac{t+1+N}{2} - t = \frac{N-t+1}{2}$.
(b) $p_T(x) = 2^{-x}$ **for** $x = 1, 2, \ldots$**:**
This is a Geometric distribution with $p = 1/2$. By the memoryless property:

$$E[T - t|T > t] = E[T] = \frac{1}{p} = 2$$

Problem 5

(a) PMF of X:
$P(X \geq k) = [P(X_i \geq k)]^3 = \left(\frac{111-k}{10}\right)^3$.

$$P(X = k) = P(X \geq k) - P(X \geq k+1) = \frac{(111-k)^3 - (110-k)^3}{1000}$$

for $k \in \{101, \ldots, 110\}$.
(b) Expected Improvement: The score X_i is uniformly distributed on $\{101, 102, \ldots, 110\}$.

$$E[X_i] = \frac{101 + 110}{2} = 105.5$$

For a non-negative integer-valued random variable X, the expectation can be calculated as $E[X] = \sum_{k=1}^{\infty} P(X \geq k)$. Since the minimum score X is at least 101, we know that $P(X \geq k) = 1$ for all $k \leq 100$. For $k > 110$, $P(X \geq k) = 0$. Thus, we can split the sum:

$$\begin{aligned} E[X] &= \sum_{k=1}^{100} P(X \geq k) + \sum_{k=101}^{110} P(X \geq k) + \sum_{k=111}^{\infty} P(X \geq k) \\ &= \sum_{k=1}^{100} (1) + \sum_{k=101}^{110} [P(X_i \geq k)]^3 + 0 \\ &= 100 + \sum_{k=101}^{110} \left(\frac{111-k}{10}\right)^3 \end{aligned}$$

Let $m = 111 - k$. As k goes from 101 to 110, m goes from 10 down to 1:

$$\begin{aligned} E[X] &= 100 + \frac{1}{1000} \sum_{m=1}^{10} m^3 \\ &= 100 + \frac{1}{1000} \left[\frac{10(11)}{2} \right]^2 \quad \text{(using } \sum m^3 = [\frac{n(n+1)}{2}]^2\text{)} \\ &= 100 + \frac{3025}{1000} = 103.025 \end{aligned}$$

$$Improvement = E[X_i] - E[X] = 105.5 - 103.025 = 2.475$$