Probability: Quiz 2

Reference Solutions

Problem 1

Proof:
Assume A and B are independent events. By the definition of independence, we have:

$$P(A \cap B) = P(A)P(B)$$

Given $P(A) = \frac{|A|}{p}$ for any $A \in \mathcal{F}$, we can substitute this into the equation:

$$\frac{|A \cap B|}{p} = \left(\frac{|A|}{p}\right)\left(\frac{|B|}{p}\right)$$

Multiplying both sides by p^2 yields:

$$p|A \cap B| = |A| \cdot |B|$$

Since $|A \cap B|$, $|A|$, and $|B|$ are all integers, the equation shows that p divides the product $|A| \cdot |B|$. Because p is a prime number, by Euclid's lemma, if a prime divides a product of two integers, it must divide at least one of those integers. Therefore, p must divide $|A|$ or p must divide $|B|$.

Without loss of generality, assume p divides $|A|$. Since $A \subseteq \Omega$ and $|\Omega| = p$, the number of elements in A is restricted by:

$$0 \leq |A| \leq p$$

The only integers in this range that are divisible by p are 0 and p.

- If $|A| = 0$, then $A = \emptyset$.
- If $|A| = p$, then $A = \Omega$.

By symmetric reasoning, if p divides $|B|$, then $B = \emptyset$ or $B = \Omega$.
Therefore, at least one of A and B must be either $\emptyset$ or Ω.

Problem 2

Let B_1 be the event that Box 1 is selected, and B_2 be the event that Box 2 is selected. Let R be the event that a red ball is picked.

Since the box is randomly selected, the prior probabilities are:

$$P(B_1) = P(B_2) = \frac{1}{2}$$

The conditional probabilities of picking a red ball from each box are:

$$P(R|B_1) = \frac{999}{1000}$$

$$P(R|B_2) = \frac{1}{1000}$$

We want to find the posterior probability $P(B_1|R)$. By Bayes' Theorem:

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)}$$

Substituting the given values into the formula:

$$P(B_1|R) = \frac{\left(\frac{999}{1000}\right)\left(\frac{1}{2}\right)}{\left(\frac{999}{1000}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{1000}\right)\left(\frac{1}{2}\right)}$$

Dividing numerator and denominator by $\frac{1}{2000}$:

$$P(B_1|R) = \frac{999}{999+1} = \frac{999}{1000} = 0.999$$

Problem 3

Let G_A be the event that Suspect A is guilty, and G_B be the event that Suspect B is guilty.
Let M be the new evidence: "Suspect A matches the blood type found at the crime scene".

Based on the initial equal evidence:

$$P(G_A) = 0.5, \quad P(G_B) = 0.5$$

If Suspect A is guilty (G_A is true), then the blood at the crime scene belongs to A. Thus, it is certain that A will match this blood type:

$$P(M|G_A) = 1$$

If Suspect B is guilty (G_B is true), then the blood at the crime scene belongs to B. The fact that Suspect A matches this blood type is merely a coincidence. Since this blood type is present in 10% of the general population, the probability of A randomly having this blood type is:

$$P(M|G_B) = 0.1$$

We need to calculate $P(G_A|M)$. Using Bayes' Theorem:

$$P(G_A|M) = \frac{P(M|G_A)P(G_A)}{P(M|G_A)P(G_A) + P(M|G_B)P(G_B)}$$

Substituting the probabilities:

$$P(G_A|M) = \frac{1 \cdot 0.5}{1 \cdot 0.5 + 0.1 \cdot 0.5}$$

$$P(G_A|M) = \frac{1}{1+0.1} = \frac{1}{1.1} = \frac{10}{11}$$

Problem 4

Let N be the total number of fuses, K be the total number of good fuses, and n be the number of fuses selected.
From the problem description, we have:

$$N = 50, \quad K = 40, \quad n = 10$$

The selection is done without replacement, which follows a hypergeometric distribution. We want to find the probability of selecting exactly $k = 10$ good fuses.

The probability is given by the ratio of the number of ways to choose 10 good fuses out of 40 to the total number of ways to choose 10 fuses out of 50:

$$P(\text{all 10 are good}) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Substituting the specific values:

$$P(\text{all 10 are good}) = \frac{\binom{40}{10}\binom{10}{0}}{\binom{50}{10}} = \frac{\binom{40}{10}}{\binom{50}{10}}$$