Probability: Quiz 3

Reference Solutions

Problem 1

Part (a):
The possible values of X are $\{-2, -1, 0, 1, 2, 3, 4\}$, each with probability $\frac{1}{7}$. The random variable $Y = |X - 1|$ maps the values of X as follows:

- $X = 1 \implies Y = 0$
- $X = 0, 2 \implies Y = 1$
- $X = -1, 3 \implies Y = 2$
- $X = -2, 4 \implies Y = 3$

By summing the probabilities of X that map to each y, the PMF of Y is:

$$
\begin{aligned}
p_Y(0) &= P(X = 1) = \frac{1}{7} \\
p_Y(1) &= P(X = 0) + P(X = 2) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7} \\
p_Y(2) &= P(X = -1) + P(X = 3) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7} \\
p_Y(3) &= P(X = -2) + P(X = 4) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}
\end{aligned}
$$

For all other values of y, $p_Y(y) = 0$.

The mean of Y, $E[Y]$, is computed by:

$$
\begin{aligned}
E[Y] &= \sum_{y=0}^{3} y \cdot p_Y(y) \\
&= 0\left(\frac{1}{7}\right) + 1\left(\frac{2}{7}\right) + 2\left(\frac{2}{7}\right) + 3\left(\frac{2}{7}\right) \\
&= \frac{0 + 2 + 4 + 6}{7} = \frac{12}{7}
\end{aligned}
$$

To find the variance of Y, we first find $E[Y^2]$:

$$\begin{aligned}
E[Y^2] &= \sum_{y=0}^{3} y^2 \cdot p_Y(y) \\
&= 0^2\left(\frac{1}{7}\right) + 1^2\left(\frac{2}{7}\right) + 2^2\left(\frac{2}{7}\right) + 3^2\left(\frac{2}{7}\right) \\
&= 0 + 1\left(\frac{2}{7}\right) + 4\left(\frac{2}{7}\right) + 9\left(\frac{2}{7}\right) \\
&= \frac{2+8+18}{7} = \frac{28}{7} = 4
\end{aligned}$$

The variance of Y is:

$$\begin{aligned}
Var(Y) &= E[Y^2] - (E[Y])^2 \\
&= 4 - \left(\frac{12}{7}\right)^2 = 4 - \frac{144}{49} \\
&= \frac{196}{49} - \frac{144}{49} = \frac{52}{49}
\end{aligned}$$

Part (b):

We are given $Z = -4|X-1| - 6$. Notice that $Z = -4Y - 6$. Using the properties of expectation:

$$\begin{aligned}
E[Z] = E[-4Y-6] &= -4E[Y] - 6 \\
&= -4\left(\frac{12}{7}\right) - 6 \\
&= -\frac{48}{7} - \frac{42}{7} = -\frac{90}{7}
\end{aligned}$$

Using the properties of variance:

$$\begin{aligned}
Var(Z) = Var(-4Y-6) &= (-4)^2 Var(Y) \\
&= 16\left(\frac{52}{49}\right) = \frac{832}{49}
\end{aligned}$$

Problem 2

The given joint probabilities $P(X = x, Y = y)$ are:

- $y = 4$: (1,4): 1/18, (2,4): 1/36, (3,4): 1/36, (4,4): 1/18
- $y = 3$: (1,3): 1/18, (2,3): 1/36, (3,3): 1/36, (4,3): 1/18
- $y = 2$: (1,2): 1/6, (2,2): 1/12, (3,2): 1/12, (4,2): 1/6
- $y = 1$: (1,1): 1/18, (2,1): 1/36, (3,1): 1/36, (4,1): 1/18

Let $Z = \min(X, Y)$. We find the probability of $Z = z$ by summing the probabilities of all pairs (x, y) such that $\min(x, y) = z$. Let us convert all fractions to a common denominator of 36 for easier summation.

For $Z = 1$, the valid pairs are $(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)$:

$$\begin{aligned} P(Z=1) &= P(1,1)+P(1,2)+P(1,3)+P(1,4)+P(2,1)+P(3,1)+P(4,1) \\ &= \frac{2}{36}+\frac{6}{36}+\frac{2}{36}+\frac{2}{36}+\frac{1}{36}+\frac{1}{36}+\frac{2}{36} \\ &= \frac{16}{36} = \frac{4}{9} \end{aligned}$$

For $Z = 2$, the valid pairs are $(2,2),(2,3),(2,4),(3,2),(4,2)$:

$$\begin{aligned} P(Z=2) &= P(2,2)+P(2,3)+P(2,4)+P(3,2)+P(4,2) \\ &= \frac{3}{36}+\frac{1}{36}+\frac{1}{36}+\frac{3}{36}+\frac{6}{36} \\ &= \frac{14}{36} = \frac{7}{18} \end{aligned}$$

For $Z = 3$, the valid pairs are $(3,3),(3,4),(4,3)$:

$$\begin{aligned} P(Z=3) &= P(3,3)+P(3,4)+P(4,3) \\ &= \frac{1}{36}+\frac{1}{36}+\frac{2}{36} \\ &= \frac{4}{36} = \frac{1}{9} \end{aligned}$$

For $Z = 4$, the only valid pair is $(4,4)$:

$$\begin{aligned} P(Z=4) &= P(4,4) \\ &= \frac{2}{36} = \frac{1}{18} \end{aligned}$$

Thus, the PMF of Z is:

$$p_Z(z) = \begin{cases} \frac{4}{9}, & z = 1 \\ \frac{7}{18}, & z = 2 \\ \frac{1}{9}, & z = 3 \\ \frac{1}{18}, & z = 4 \\ 0, & \text{otherwise} \end{cases}$$

Problem 3

Solution:

Part (a):

Let $X \sim \text{Bernoulli}(p)$ where $p = 1/2$.

$$\text{Mean} = E[X] = p = \frac{1}{2}$$

$$\text{Variance} = Var(X) = p(1-p) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

Part (b):
Let $Y \sim \text{Binomial}(n, p)$ where $p = 1/2$.

$$\text{Mean} = E[Y] = np = n\left(\frac{1}{2}\right) = \frac{n}{2}$$

Problem 4

Solution:
Part (a):
A "Discrete random variable" is a random variable whose range (the set of all possible values it can take) is finite or countably infinite.
Part (b):
For a Poisson random variable $X \sim \text{Poisson}(\lambda)$, the expected value is $E[X] = \lambda$. Given $\lambda = 1/2$:

$$\text{Mean} = \frac{1}{2}$$

Part (c): For a Geometric random variable $Y \sim \text{Geometric}(p)$ (defined as the number of trials until the first success), the expected value is $E[Y] = 1/p$. Given $p = 1/2$:

$$\text{Mean} = \frac{1}{1/2} = 2$$